# Pre-class Warm-up

Do you remember what the gradient of a function is?

Can we do grad f if

No.

a. f : R -> R^2 ? e.g. f(t) = (t,t^2) ← Another way to ask this: find grad f.

b. f : R^2 -> R ? e.g. f(s,t) = s + t^2 Yes

c. f : R^3 -> R ? e.g. f(x,y,z) = xy + z

Second question: What is the difference between grad f and the derivative Df

a. Pretty much the same thing

b. Some similarities, but different.

c. Completely different

Third question: would you prefer it if I made the font a bit bigger?

1. Yes

2. No

b. grad $f = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$

$= (1, 2t)$

$Df = [1, 2t]$

$Df : \mathbb{R}^2 \to \mathbb{R}$

$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x + 2t \cdot y$

## Section 2.6: Directional derivatives and the gradient

We learn:

- What is the directional derivative of a function f : R^3 -> R?
  (It could be f: R^n -> R )

- The connection between the gradient and the directional derivative.

- The gradient points in the direction of greatest increase of f .

- The gradient points perpendicular to level sets.

- Using this to compute tangent planes etc.

## The directional derivative

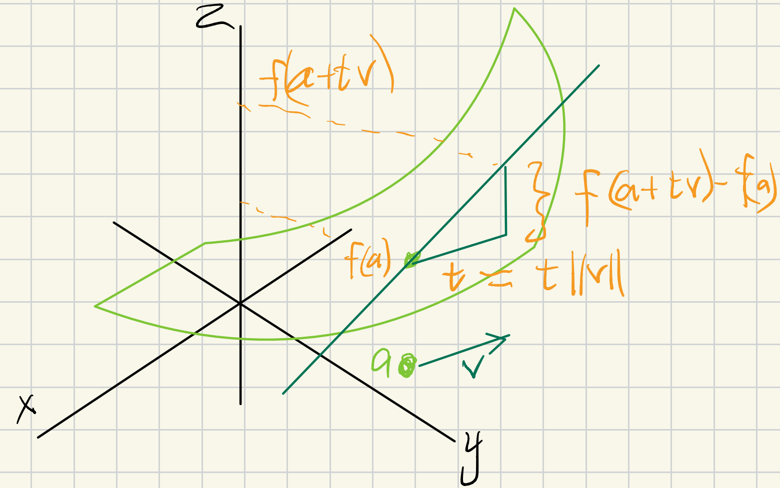
Suppose we have a function f : R^n -> R.
Let v be a vector of length 1 and a any vector in R^n .
The directional derivative of f at a in the direction v is

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

What if v wasn't a unit vector?

We get confused.

The book sticks to n = 3. When n = 2 we can draw the graph of f :




The directional derivative is the slope of the graph in direction v .

Theorem 12 Let f : R^n -> R, a and v vectors in R^n with v of length 1.
The directional derivative equals

matrix multn

Df(a) v = grad f (a) • v

useful notation

If these are written out fully it looks like:

$$\left.\frac{\partial f}{\partial x}\right|_a v_1 + \left.\frac{\partial f}{\partial y}\right|_a v_2 + \left.\frac{\partial f}{\partial z}\right|_a v_3$$

Proof. We can use the chain rule.

Let $c(t) = a + tv \in \mathbb{R}^n$.

The directional derivative is

$$\left.\frac{d}{dt} f(c(t))\right|_0 = Df(c(0)) \cdot Dc(0)$$

$$= \left[\left.\frac{\partial f}{\partial x}\right|_a \quad \left.\frac{\partial f}{\partial y}\right|_a \quad \left.\frac{\partial f}{\partial z}\right|_a\right] c'(t)$$

$$= Df(a) \cdot v$$

Example: Compute the directional derivative of f(x,y) = x^2 + xy at $a = (1, 2)$
in the direction of (3/5,4/5).

$$\nabla f = (2x+y, x)$$

The direction derivative is

$$\nabla f \cdot (3/5 \quad 4/5) = \frac{3(2x+y)}{5} + \frac{4x}{5}$$

evaluated at $(x,y) = (1,2)$.

Quick question:

Is $\partial f / \partial x$ any of the following?

a. a unit vector

b. a directional derivative, in direction $x$ ✓

c. a directional derivative, in direction $y$

Theorem 13 If grad f (a) $\neq 0$ then grad f (a) points in the direction along which f is increasing the fastest.

Proof. In any unit direction v, f increases at rate $\nabla f(a) \cdot v$ $= \|\nabla f(a)\| \cdot \|v\| \cos\theta$ where $\theta$ = angle between $\nabla f(a)$ and v. This is largest when $\theta = 0$, v points in direction $\nabla f(a)$.

Theorem 14 If S is a level set of f defined by $f(a) = k$ then grad f (a) is perpendicular to S.

Proof. If v points tangentially to the level set then the slope of f in that direction is 0. Thus $\nabla f(a) \cdot v = 0$, $\nabla f(a)$ is perpendicular to v.

This means we can compute tangent planes to surfaces, because grad f is a normal vector

Example. Compute the tangent plane to the surface x^2 + y^2 + z = 7 at the point (2,1,2).

Solution. The surface is a level set of the function $f(x,y,z) = x^2 + y^2 + z$

$\nabla f = (2x \quad 2y \quad 1) = (4 \quad 2 \quad 1)$ at $(2 \; 1 \; 2)$

The plane has equation

$4x + 2y + z = D$.

Substitute $(x,y,z) = (2,1,2)$

to get $8 + 2 + 2 = D = 12$

Answ. $4x + 2y + z = 12$

Like qn 4. You are walking on the graph of f(x,y) = xy^2 + y + 3 standing at the point (2,1,6). Find an (x,y)-direction you should walk in to stay at the same level.

Solution. $\nabla f = (y^2, 2xy+1)$

When $(x,y) = (2,1)$

$\nabla f(2,1) = (1, 5)$.

This is perp. to the level direction.

Solve $(v_1, v_2) \cdot (1, 5) = 0$

Answ $(5, -1)$